Nonvanishing Results for Kähler Varieties

j.w. A. Höring, V. Lazić

Thm 1 (X, Δ) $\mathbb{Q}$-factorial compact Kähler klt pair s.th. X not uniruled. If $\mathrm{nd}(K_X + \Delta) = 1$, $K_X + \Delta$ nef, $\chi(\mathcal{O}_X) \neq 0$, then $\kappa(K_X + \Delta) \geq 0$.

Thm 2 X hyperkähler mfd, $\dim X = 2n$, L nef line bundle, $q_X(L) = 0$, Then $\kappa(L) \geq 0$ or

- $\forall T \in c_1(L)$ closed pos. current and $\forall Z \subset X$ max Lelong comp. we have: (i) $n \leq \dim Z \leq 2n-2$

(ii) Z is coisotropic wrt hol. sympl form

(iii) $c_1(L)|_Z^{\dim Z - n + 1} = 0$

Recall:

- currents $= \begin{pmatrix} \text{differential} \\ \text{forms} \end{pmatrix}^\vee$

- X smooth $\Rightarrow$ $\exists$ dR thm for currents

- $\alpha \in H^{1,1}(X, \mathbb{R})$ pseff

  $:\Longleftrightarrow$ $\exists\, T \in \alpha$ closed positive

Demailly: L pseff $\Leftrightarrow$ $c_1(L)$ pseff

- (V^{2n}, σ) sympl. vector space

  $W \subset V$ isotropic $:\Leftrightarrow$ $W \subset W^\perp$

  coisotropic $:\Leftrightarrow$ $W^\perp \subset W$

  $(\Rightarrow n \le \dim W \le 2n)$

§1 Preliminaries

T closed positive current, $x \in X$

$\rightsquigarrow$ $\nu(T,x) \geq 0$

Lelong number of T at x

$\rightsquigarrow$ $Z \subset X$ subvar

$\rightsquigarrow$ $[Z]$ = current of integration

Theorem (Siu) $c > 0$, then

$E_c(T) := \{ x \in X \mid \nu(T,x) \geq c \}$

$\subsetneq X$ is analytic.

<u>Defn</u> max. Lelong component

$:=$ irred. component of $E_c(T)$ of

max. dim $\forall c > 0$, $c \leq$ some c_0

Thm (Demailly) α nef, $Z \subset X$ max Lelong component, $\operatorname{codim} Z = p$, $S \in \alpha$ pos current

$\Rightarrow \exists\, T \geq 0$, $T \in \alpha^p$ s.th.

$$T - \underline{\underline{\nu(S, Z)}}^p\, [Z] \geq 0$$

§2 Good bilinear forms

Defn X cp. Kähler, A good bil. form on X is

- symmetric bil. form

$$q : H^{1,1}(X, \mathbb{R}) \times H^{1,1}(X, \mathbb{R}) \to \mathbb{R}$$

s.th. $\quad \operatorname{sign}(q) = (1, h^{1,1} - 1)$

and $\quad q(\alpha, \beta) \geq 0 \quad \forall\, \beta$ pseff

α "modified nef"

Prop: (1) X cp Kähler, γ Kähler class

$$\Rightarrow \quad q(\alpha, \beta) := \int_X \alpha . \beta . \gamma^{n-2}$$

is good

(2) X HK mfd

q_X = BBF form on $H^2(X, \mathbb{Z})$

$$\Rightarrow \quad q_X \big|_{H^{1,1}(X, \mathbb{R})} \text{ is good.}$$

Thm 6.4 X cp Kähler, q good, α modified nef, $q(\alpha) = 0$. Assume $\exists\, \beta$ pseff, $D > 0$ $\mathbb{R}$-div on X s.th. $\alpha = \beta + \{D\}$.

Then $\exists\, E \geq 0$: $\alpha = \{E\}$

Pf: Boucksom – Zariski decomp

$$\beta \ni T = \underset{\substack{\uparrow \\ \text{modified} \\ \text{nef}}}{S} + \underset{\substack{\uparrow \\ \text{eff. div}}}{N}$$

$$\alpha = \underbrace{\{S\}}_{=:R} + \{N + D\}$$

$$0 = q(\alpha) = \underbrace{q(\alpha, R)}_{\geq 0} + \underbrace{q(\alpha, N + D)}_{\geq 0}$$

$$\Rightarrow \quad q(\alpha) = 0, \quad q(\alpha, R) = 0$$

$$q(R) \geq 0 \qquad \text{Assume } q(R) > 0$$

$$\lightning \text{ to } q(\alpha) = 0$$

$$\Rightarrow \quad q(R) = 0$$

$$\Rightarrow \quad \exists\, t > 0 : \quad R = t \cdot \alpha, \quad t \neq 1$$

$$\Rightarrow \quad E := \frac{1}{1-t}(D + N)$$

∎

proof of Thm 2:

Let $T \in \underset{\text{nef}}{\alpha} := c_1(L)$ closed pos. current

If all Lelong numbers of T vanish,

then $\kappa(L) \geq 0$ by Verbitsky.

Otherwise:

Let $Z \subset X$ be max. Lelong component

suppose Z divisor:

$$T \overset{\text{Demailly}}{=} \underbrace{(T - \nu \cdot [Z])}_{\geq 0} + \nu \cdot [Z]$$

$$\Big\downarrow \{\cdot\}$$

$$\alpha = \underset{\text{pseff}}{\beta} + \underset{\text{eff.}}{\{\nu [Z]\}}$$

$$\Rightarrow \alpha \overset{6.4}{=} \{E\}$$

X HK $\Rightarrow H^1(\mathcal{O}_X) = 0 \Rightarrow \kappa(L) \geq 0.$

remains to be shown:

Z max Lelong comp $\dim Z \leq 2n-2$

Then (i) Z coisotropic

(ii) $\alpha|_Z^{n+1-p} = 0$

Pf of (ii):

property of BBF form q_X:

$$q_X(\alpha) = 0 \implies \alpha^{n+1} = 0$$

$$\alpha \neq 0 \qquad\qquad \alpha^n \neq 0$$

$$0 = \alpha^{n+1} \cdot \eta^{n-1} = \alpha^{n-p+1} \{ \underbrace{S}_{\substack{\geq 0 \\ \text{current}}} + \nu \cdot [Z] \} \eta^{n-1}$$

η Kähler class

$$\implies \{[Z]\} \cdot \alpha^{n-p+1} \cdot \eta^{n-1} = 0$$

$$\Rightarrow \quad \alpha\big|_Z^{n-p+1} = 0$$

Z coisotropic: $\quad \sigma$ hol. sympl. form

$$\alpha^p \cdot \{\sigma \wedge \bar{\sigma}\}^{n-p+1} \cdot \eta^{p-2} = 0$$

similarly

$$\Rightarrow \quad \sigma \wedge \bar{\sigma}^{\,n-p+1}\big|_{Z^{reg}} = 0$$

$$\Rightarrow \quad \sigma^{n-p+1}\big|_{Z^{reg}} = 0 .$$

$\square$